Journal of Statistical Physics, Vol. 30, No. 3, 1983

Approximate Gaussian Representation of Evolution Equations I. Single Degree of Freedom Nonlinear Equations

Bruce J. West,[1] Galina Rovner,[1] and Katja Lindenberg[2]

Received September 10, 1982

We have developed a generalization of the method of statistical linearization to enable us to describe transient and other nonstationary phenomena obeying stochastic nonlinear differential equations. This approximation technique provides an optimal Gaussian representation with time-dependent parameters. The algorithm specifies a set of ordinary differential equations for the Gaussian parameters in terms of the time-dependent average nonlinearities. We apply the general formalism developed herein for single degree of freedom dissipative systems to a particular example.

KEY WORDS: Stochastic; nonlinear; approximation; optimum.

1. INTRODUCTION

Many problems arising in the study of physical phenomena are most naturally represented in terms of systems of nonlinear stochastic differential equations.[1-4] The difficulty in obtaining solutions to such equations is twofold. The first is the proper treatment of the nonlinear aspects of the system, and the second is the analysis of the stochastic behavior. Several approximation techniques have evolved for the analysis of nonlinear stochastic equations. Among the most useful of these is the method of statistical linearization.[5-16]

Supported by the Defense Advanced Research Projects Agency DARPA-0053.

[1] Center for Studies of Nonlinear Dynamics, La Jolla Institute, P.O. Box 1434, La Jolla, California 92038.

[2] Department of Chemistry, University of California at San Diego, La Jolla, California 92093.

0022-4715/83/0300-0633$03.00/0

The method of statistical linearization consists of the replacement of nonlinear stochastic equations with linear ones whose coefficients are determined by an error minimization algorithm. This algorithm is appropriate in the asymptotic (i.e., stationary) regime and leads to algebraic equations for the coefficients of the "equivalent" linear system.

In many physical systems it is the transient regime in which one is interested.[1-17] In other physical situations one is interested in driven systems that never achieve a time-independent stationary state.[18,19] One example that has been of considerable recent interest is that of the transient radiation of a laser close to threshold.[17] Another is the water wave field driven by the wind.[16] These time-dependent features of the evolution cannot be treated faithfully by ordinary statistical linearization.

We have extended the method of ordinary statistical linearization to allow for the description of such time-dependent phenomena. This has been accomplished by allowing the variational coefficients of the "equivalent" linear system to be time dependent, and minimizing the corresponding time-dependent error at each time. Instead of algebraic equations we now obtain ordinary differential equations for the linearization parameters. These evolution equations ensure that the algorithm minimizes the error continuously as the system evolves in time. We have therefore replaced the original system of stochastic nonlinear differential equations (with constant coefficients) with a system of linear stochastic equations with time-dependent coefficients. As noted, these time-dependent coefficients are determined by a set of ordinary differential equations. Related techniques have been developed by Eaves and Reinhardt[20] and also by Iwan and Mason.[21]

The difficulty in determining the statistical properties of the system by solving the stochastic dynamic equations directly is well known: the calculation necessitates generating an ensemble of fluctuations and generating a solution of the equations for each member of the ensemble. This is a tractable method for linear systems but practically impossible for most nonlinear systems. A more viable procedure (even for linear systems) is to consider instead the time evolution of the phase space distribution rather than the dynamical equations. Of course the two descriptions are formally equivalent, as can be seen from Liouville's theorem.[2] The phase space distribution for a linear system with Gaussian delta-correlated fluctuations follows immediately from a generalization of Doob's theorem[22] and is a multivariate Gaussian. A linear system with n degrees of freedom can be described by a Gaussian with an $n \times n$ correlation matrix and n means, all of which are generally time dependent. We can relate these distribution parameters to the time-dependent linearization coefficients in our "equivalent" linear system. We have therefore replaced an ensemble of

nonlinear systems of unknown statistics by an ensemble of linear systems with Gaussian (time-dependent) statistics.

In this paper we restrict our analysis to single degree of freedom systems. The formal procedure for extending the method to more degrees of freedom is straightforward. However, the algebraic details are sufficiently tedious that one would carry them out only for particular applications rather than in full generality. This we do for the transient radiation of a single mode laser in the sequel.[23]

In Section 2 we present our method for a general dissipative single degree of freedom system. In Section 3 we apply the procedure to a particular system, i.e., one with cubic and quadratic nonlinearity. We consider both initial condition effects as well as the time-dependent behavior due to a harmonic driver. We end with a short conclusion in Section 4.

2. APPROXIMATE GAUSSIAN REPRESENTATION OF EVOLUTION EQUATIONS (AGREE)

In this section we investigate the optimal linear representation of fluctuations for a single degree of freedom nonlinear system. It is our intent to take into account the full nonlinear character of the system in the absence of fluctuations, but to consider the optimal linearization of fluctuations about the nonlinear deterministic behavior of the system.

Consider the single degree of freedom stochastic differential equation

$$\dot{x} + \lambda x + F(x) = Q(t) + M(t)f(t) \tag{2.1}$$

In Eq. (2.1) λ is a real parameter, $F(x)$ is a nonlinear function of x, $Q(t)$ is a deterministic driver, $f(t)$ is a stochastic function, and $M(t)$ is a deterministic modulation of the stochastic driver. We assume that the fluctuations have the following properties: (1) They are zero-centered,

$$\langle f(t) \rangle = 0 \tag{2.2}$$

(2) They are delta correlated in time with strength $2D$,

$$\langle f(t)f(t') \rangle = 2D\delta(t - t') \tag{2.3}$$

and (3) the higher cumulants vanish, i.e., $f(t)$ has Gaussian statistics.

We separate the evolution of the system into the deterministic portion $y(t)$ that describes the system in the absence of the fluctuating driver, and a portion $\zeta(t)$ that describes the fluctuating dynamics of the actual system about the deterministic path. We thus write

$$x(t) = y(t) + \zeta(t) \tag{2.4}$$

where $y(t)$ is the solution of (2.1) in the absence of fluctuations and hence

obeys the deterministic equation

$$\dot{y}(t) + \lambda y(t) + F(y) = Q(t) \tag{2.5}$$

with initial condition $y(0) = x(0)$. The difference variable is therefore governed by the stochastic equation of motion obtained by subtracting (2.5) from (2.1), i.e.,

$$\dot{\zeta}(t) + \lambda\zeta(t) + G(y,\zeta) = M(t)f(t) \tag{2.6}$$

with

$$G(y,\zeta) = F(y+\zeta) - F(y) \tag{2.7}$$

and with the initial condition $\zeta(0) = 0$. We note that (2.5) and (2.6) still constitute an *exact* representation of the system (2.1).

2.1. Time-Dependent Generalization of Statistical Linearization

We now wish to replace (2.6) with an optimal approximate linear stochastic differential equation. This procedure for treating the stochastic variable $\zeta(t)$ approximately while retaining the exact solution for the deterministic portion $y(t)$ is sensible if the principal variations of the functions $y(t)$ and $\zeta(t)$ occur on different time scales. To clarify what is meant by this statement, let us first consider equation (2.6). The temporal variations of $\zeta(t)$ are determined by the parameter λ, by the variations of $y(t)$ (cf. below) and $M(t)$, by the form and strength of the nonlinear function $G(y,\zeta)$, and by the stochastic force $f(t)$. We presume that of these, the shortest time scale is introduced by the stochastic fluctuation $f(t)$ [otherwise the entire description (2.1) with the assumption (2.3) is unreasonable], and that no important features on this short time scale other than those produced by $f(t)$ occur in (2.6). We thus presume that $\dot{\zeta}(t)$ fluctuates rapidly about a slowly varying path, and that these two variations of $\dot{\zeta}(t)$ are quite distinct.

Since the stochastic function $\zeta(t)$ depends on $y(t)$ via the nonlinear function $G(y,\zeta)$, implicit in the above statement is the assumption that the temporal variations of $y(t)$ are much slower than those of $f(t)$. The time scales for variations of $y(t)$ are determined by the parameter λ, the period of the driving function $Q(t)$, and the nonlinear function $F(y)$ in (2.5). Although such a nonlinear term in general can generate higher and higher frequencies in $y(t)$ as the system evolves, we must restrict our studies to cases in which these high-frequency components either contribute negligibly to $y(t)$ or, at least, do not seriously affect the behavior of $\zeta(t)$. In other words, we restrict our method to systems in which the principal variation of y with time is slow compared to the fluctuations introduced by $f(t)$.

Since we specifically wish to consider nonstationary problems [i.e., since the function y occurring in (2.6) depends on time], it might be

expected that the linear equation that best reproduces the behavior of (2.6) (by any measure one wishes to choose) also changes with time. It then becomes appropriate to replace (2.6) by a different linear equation at each time point, i.e., by a linear equation with time-dependent coefficients. In this paper we develop a procedure to carry out just such an optimization at each time.

The method we develop below is a suitable generalization of the method of statistical linearization. In the past, statistical linearization has been developed for the study of *stationary* properties of nonlinear stochastic differential equations of the form (2.1) with $Q(t) = 0$ and $M(t) = 1$, and is based on the replacement of (2.1) as $t \to \infty$ with the linear equation[5–16]

$$\dot{x} + (\lambda + \mu)x = f(t) \tag{2.8}$$

This replacement is suitable if $F(x) + \lambda x$ is the derivative of an open concave function of x, and we restrict ourselves to this case in our discussion of the traditional method. The parameter μ is chosen so as to minimize the mean square value of the error $\epsilon(x) = [F(x) - \mu x]$ made in Eq. (2.1) by the replacement of $F(x)$ by μx. The expression for μ that emerges from this criterion is

$$\mu = \frac{\langle xF(x)\rangle}{\langle x^2\rangle} \tag{2.9}$$

The averages in (2.9) are taken over an ensemble of realizations of $f(t)$ as $t \to \infty$. Alternatively, one can construct an evolution equation for the phase space probability density $P(x,t)$ of the stochastic process; the averages in (2.9) are then calculated with respect to the stationary distribution $P(x, t \to \infty)$ [the existence of this distribution is ensured by the form of $F(x)$ assumed here], e.g.,

$$\langle xF(x)\rangle = \int dx\, P(x, t \to \infty) xF(x) \tag{2.10}$$

The value of μ obtained from (2.9) is therefore *constant* in time. It can easily be shown that the mean value $\langle x\rangle = 0$ obtained from (2.8) as $t \to \infty$ is identical with the mean value obtained from (2.1) when $Q = 0$, $M = 1$ and $t \to \infty$. The mean square value $\langle x^2\rangle$ obtained from (2.8) is identical with that of (2.1) if the exact distribution $P(x, t \to \infty)$ is used in the calculation of μ. If the distribution corresponding to (2.8) is used in (2.9) so that μ is determined self-consistently, then the mean square value obtained from (2.8) is approximately equal to that of the nonlinear system. In Fig. 1 we compare the exact vs. approximate mean square values as a function of α for the choice $F(x) = \alpha x^3$ for fixed parameters λ and D. In Fig. 1 we show the exact "potential function"

$$V(x) = \int_0^x F(x')\, dx' + \frac{\lambda}{2} x^2 \tag{2.11}$$

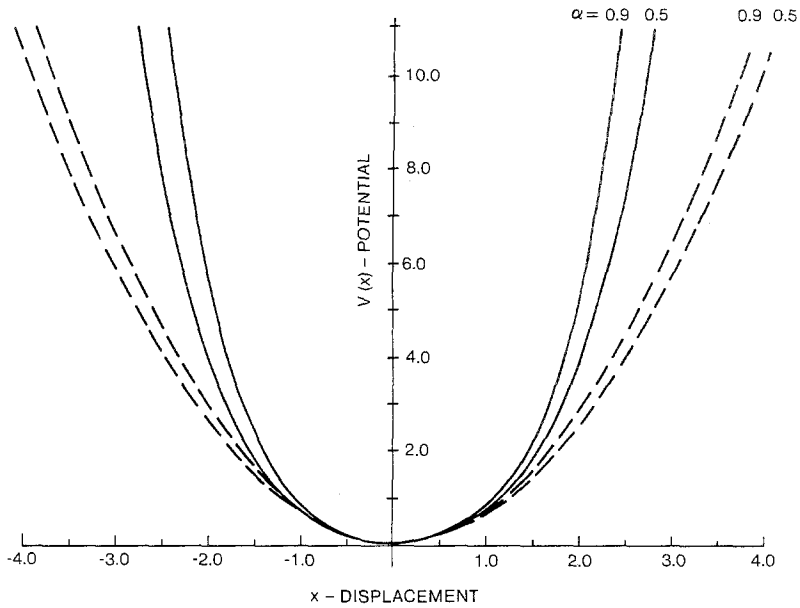


Fig. 1. The potential function for exact —— and SL - - - for two values of α.

for the example $V(x) = \alpha x^4/4 + \lambda x^2/2$ and we also show the harmonic potential function $(\lambda + \mu)x^2/2$ for two values of α, with μ obtained self-consistently. The harmonic potential is clearly designed to approximate the exact one in the region $x^2 \lesssim \langle x^2 \rangle$.

We wish to generalize the method of statistical linearization to nonstationary solutions of (2.1). The nonstationarity may be due to transient behavior from a given initial condition and/or it may be due to the nonstationary functions $Q(t)$ and $M(t)$. The generalization will therefore allow us to analyze (2.1) for all times. The long-time correlation properties for a system with $Q(t) = 0$ and $M(t)$ not constant have been analyzed in Ref. 21. To carry out our generalization, we consider (2.1) in the representation (2.5) and (2.6), and then replace (2.6) by the approximate linear equation

$$\dot{\zeta}(t) + [\lambda + \mu(t)]\zeta(t) - \gamma(t) = M(t)f(t) \tag{2.12}$$

where $\mu(t)$ and $\gamma(t)$ are time-dependent functions to be determined. The error made in the equation by this replacement is

$$\epsilon(t) = [G(y, \zeta) - \mu(t)\zeta(t) + \gamma(t)] \tag{2.13}$$

The functions $\mu(t)$ and $\gamma(t)$ are chosen so as to minimize the mean square

value of the error $\epsilon(t)$ *at each point in time*, i.e., by requiring that

$$\frac{\delta\langle\epsilon^2(t)\rangle_t}{\delta\mu(t)} = 0, \qquad \frac{\delta\langle\epsilon^2(t)\rangle_t}{\delta\gamma(t)} = 0 \tag{2.14}$$

Substituting (2.13) into (2.14) and solving explicitly for $\mu(t)$ and $\gamma(t)$ gives the relations

$$\mu(t) = \frac{\langle\zeta G(y,\zeta)\rangle_t - \langle\zeta\rangle_t\langle G(y,\zeta)\rangle_t}{\langle\zeta^2\rangle_t - \langle\zeta\rangle_t^2} \tag{2.15}$$

$$\gamma(t) = \frac{\langle\zeta\rangle_t\langle\zeta G(y,\zeta)\rangle_t - \langle\zeta^2\rangle_t\langle G(y,\zeta)\rangle_t}{\langle\zeta^2\rangle_t - \langle\zeta\rangle_t^2} \tag{2.16}$$

To complete the prescription for the determination of the time-dependent linearization functions, it is necessary to specify how the averages $\langle\ \rangle_t$ are to be evaluated. The averages are still to be interpreted as averages of the dynamical variables over an ensemble of realizations of the fluctuations, but now at time t rather than as $t\to\infty$. Equivalently, one can again construct an evolution equation for the phase space density $P(x,t)$ and compute averages according to the definition

$$\langle G(x(t))\rangle_t \equiv \int dx\, G(x)P(x,t) \tag{2.17}$$

Ideally one would use the exact probability density in (2.17). However, that implies that one can solve the original problem exactly. Since this is what one cannot do, it is necessary to use an approximate probability density. We propose to use the exact probability density obtained for the linearized problem (2.12). Since this density contains the unknown functions $\mu(t)$ and $\gamma(t)$, equations (2.15) and (2.16) constitute self-consistent integral equations for the unknown functions. As $t\to\infty$ and $\mu(t)$ and $\gamma(t)$ become constants, $P(x,t)$ approaches the equilibrium distribution appropriate to ordinary statistical linearization and the integral equations (2.15) and (2.16) then reduce to algebraic equations.

2.2. Approximate Gaussian Distribution

The equation of evolution for $P(\zeta,t)$ corresponding to (2.12) is the Fokker–Planck equation

$$\frac{\partial}{\partial t}P(\zeta,t) = \frac{\partial}{\partial\zeta}\left[\{[\lambda+\mu(t)]\zeta - \gamma(t)\}P(\zeta,t)\right] + DM^2(t)\frac{\partial^2}{\partial\zeta^2}P(\zeta,t) \tag{2.18}$$

Since Eq. (2.12) is linear, the solution of (2.18) must be Gaussian with time-varying mean $s(t)$ and variance $v(t)$:

$$P(\zeta,t)=\left[2\pi v(t)\right]^{-1/2}\exp\left\{-\left[\zeta-s(t)\right]^2/2v(t)\right\} \tag{2.19}$$

The functions $s(t)$ and $v(t)$ must of course be uniquely determined by the minimization functions $\gamma(t)$ and $\mu(t)$. The relation between $s(t), v(t)$ and $\gamma(t)$, $\mu(t)$ is not a trivial one and can only be given directly through integral relationships or through differential relationships. It is more straightforward to construct the differential relationships by direct application of the Fokker–Planck equation (2.18). The derivative of the mean value of ζ is

$$\begin{aligned}\dot{s}(t)&=\int\zeta\frac{\partial}{\partial t}P(\zeta,t)\,d\zeta\\&=-\left[\lambda+\mu(t)\right]s(t)+\gamma(t)\end{aligned} \tag{2.20}$$

where we have used the right-hand side of (2.18). It is clear that (2.20) could also have been obtained by direct averaging of the linearized equation (2.12). The derivative of the variance is

$$\begin{aligned}\dot{v}(t)&=\int\zeta^2\frac{\partial}{\partial t}P(\zeta,t)\,d\zeta-2s(t)\dot{s}(t)\\&=-2\left[\lambda+\mu(t)\right]v(t)+2DM^2(t)\end{aligned} \tag{2.21}$$

where again we have used the right-hand side of (2.18). This equation cannot be obtained directly by multiplying (2.12) by $\zeta(t)$ and then averaging since then one must determine the correlation $\langle\zeta(t)f(t)\rangle$.

One can now in principle solve (2.20) and (2.21) for $s(t)$ and $v(t)$ with $s(0)=v(0)=0$ and use this solution to express the right-hand sides of (2.15) and (2.16) solely in terms of $\gamma(t)$ and $\mu(t)$. Equivalently and more conveniently for computational purposes, we can use (2.15) and (2.16) to express $\gamma(t)$ and $\mu(t)$ in terms of $s(t)$ and $v(t)$, and use these results to close the system of equations (2.20) and (2.21).

We have thus managed to approximate the nonstationary probability distribution corresponding to a nonlinear stochastic differential equation (2.1) by a Gaussian distribution with time-dependent parameters that satisfy a set of coupled deterministic differential equations. We refer to this reduction technique as AGREE. The only exact ways to solve the problem (2.1) are either by direct integration involving an ensemble of N members, where N must be large in order to get adequate statistics, or by the solution of the exact Fokker–Planck equation, which is a partial differential equa-

tion with nonlinear coefficients. Either of these is substantially more arduous than the numerical solution of the small set of first-order ordinary differential equations involved in AGREE.

2.3. Limitations on the Present Version of AGREE

Since the exact distribution corresponding to (2.1) is here approximated by the unimodal distribution (2.19), we expect the approximation to be adequate when the potential function

$$V(\zeta; y) = \int_0^{\zeta} G(\zeta', y)\, d\zeta' + \frac{\lambda \zeta^2}{2} \tag{2.22}$$

has a single minimum at all times. For potentials with more than one minimum, it may be necessary to extend the present formalism to multimodal distributions.

3. APPLICATION

The test of any approximation technique is how faithfully the approximate solution reproduces the exact solution. For stochastic differential equations a single realization of the solution does not have any special significance. Only the moments of the solution correspond to physically interesting quantities. We select the equation

$$\dot{x}(t) + \lambda x(t) + Bx^3(t) + Ax^2(t)\operatorname{sgn} x(t) = f(t) + Q(t) \tag{3.1}$$

as the model system on which to test the AGREE. This approximation is compared with the exact solution obtained by numerically integrating (3.1) for each of an ensemble of realizations of the fluctuations $f(t)$. The ensemble of solutions is used to calculate $\langle x \rangle_t$ and $\langle x^2 \rangle_t - \langle x \rangle_t^2$, which can then be compared with $s(t)$ and $v(t)$ calculated using AGREE.

To implement the linearization technique we must evaluate the time-dependent parameters $s(t)$ and $v(t)$ using the specific nonlinear interaction from (3.1), i.e.,

$$F(x) = Bx^3 + Ax^2 \operatorname{sgn} x \tag{3.2}$$

From (2.15) and (2.16) we see that the necessary averages are $\langle \zeta G(\zeta, y) \rangle_t$, $\langle G(\zeta, y) \rangle_t$ and the moments of the deviation $\zeta(t)$ from the deterministic motion. Using (3.2) all these averages reduce to the two basic forms $\langle \zeta^n \rangle_t$ and $\langle \zeta^n \operatorname{sgn} \zeta \rangle_t$, $n = 0, 1, 2, 3$ and 4 taken with respect to the probability

density (2.19). The resulting rate equations for the mean and variance are

$$\dot{s}(t) + \lambda s(t) = -B\left\{3\left[v(t) + y(t)s(t)\right]\left[s(t) + y(t)\right] + s^3(t)\right\}$$
$$-A\left\{2y(t)\left[v(t)/2\pi\right]^{1/2}\exp\left[-(y(t) + s(t))^2/2v(t)\right]\right.$$
$$+\left[(s(t) + y(t))^2 + v(t)\right]\operatorname{erf}(y(t) + s(t))$$
$$\left. - y^2(t)\operatorname{sgn} y(t)\right\} \tag{3.3}$$

and

$$\dot{v}(t) + 2\lambda v(t) = 2D - 2B\left\{3v(t)\left[v(t) + (s(t) + y(t))^2\right]\right\}$$
$$-A\left\{\left[2v(t)/\pi\right]^{1/2}\left[s^2(t) - 2y(t)s(t) + 2v(t)\right]\right.$$
$$\times \exp\left[-(y(t) + s(t))^2/2v(t)\right]$$
$$\left. + 2v(t)\left[s(t) + y(t)\right]\operatorname{erf}(y(t) + s(t))\right\} \tag{3.4}$$

which are solved subject to the initial conditions $s(0) = v(0) = 0$. Equation (3.3), and (3.4) along with

$$\dot{y}(t) + \lambda y(t) = -By^3(t) - Ay^2(t)\operatorname{sgn} y(t) + Q(t) \tag{3.5}$$

constitute the approximate description AGREE of the stochastic differential equation (3.1). We solve (3.5) subject to the initial condition $y(0) = \text{const}$.

In Fig. 2 we compare the first two moments obtained from AGREE with the exact solution of this model system with $Q(t) = 0$. In this and subsequent comparisons it should be noted that the smooth curve is the AGREE solution while the jagged curve is the "exact" solution obtained as an ensemble average. The "exact" solution would be truly exact (and smooth) if there were an infinite number of members in the ensemble. We see that as the number of members of the ensemble is increased from 100 to 400, the irregularities in the exact moments are sufficiently reduced that it becomes clear that the "exact" solution is converging to the AGREE solution.

We observe that the solution to the model system for the values of the parameters selected in Fig. 2 decreases monotonically and the variance of the distribution increases rapidly at early times and appears to be reaching some asymptotic constant value. This solution as $t \to \infty$ yields a zero mean value with a finite variance provided by the fluctuating driving force. The transient behavior is very much like that observed in Brownian motion, i.e., the net response is dissipative. We have examined these moments when the

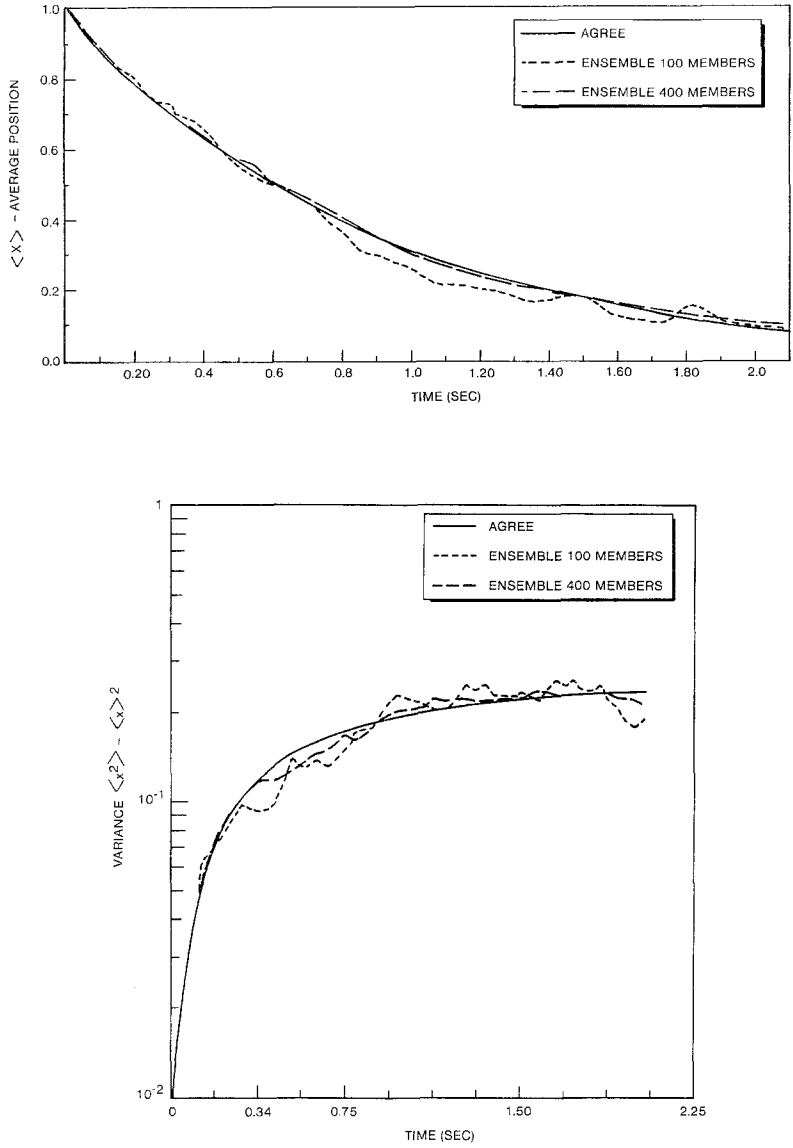
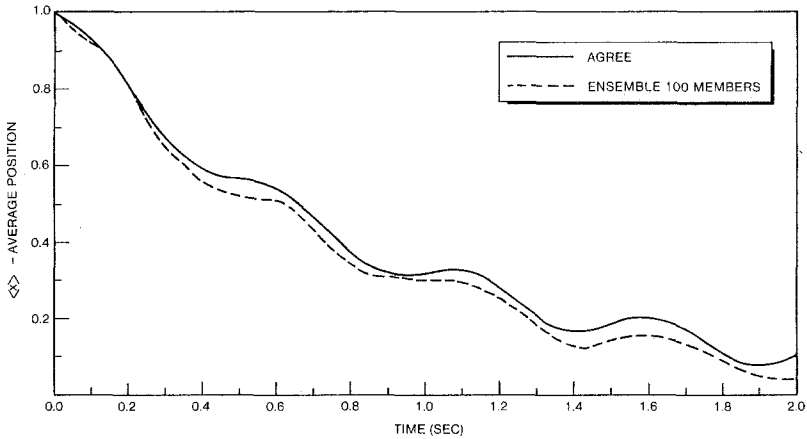


Fig. 2. The average oscillator displacement (a) and variance (b) are depicted as functions of time using AGREE and direct integration of an ensemble of (3.1) with $Q(t) = 0$; $A = B = 0.1$, $D = 0.25$, $\lambda = 1.0$.



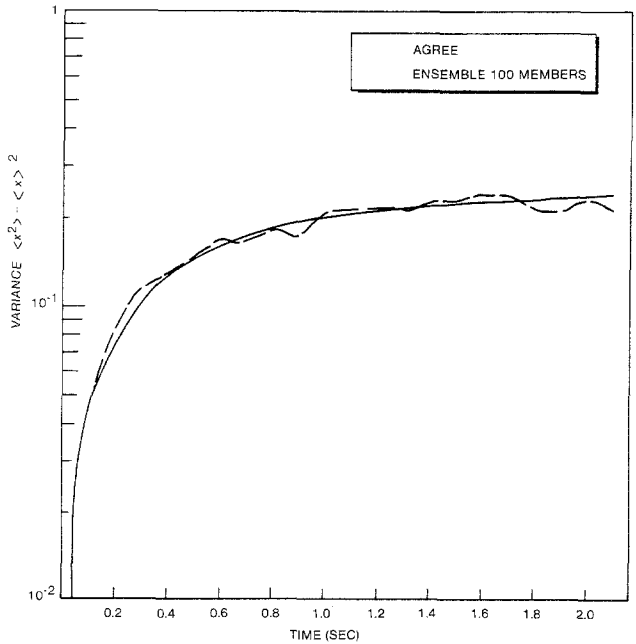


Fig. 3. The average oscillator displacement (a) and variance (b) are depicted as functions of time using AGREE and direct integration of an ensemble of (3.1) with $Q(t) = A_1 \cos \Omega t$; $A_1 = 0.5$, $\Omega = 4\pi\lambda$, $A, B = 0.1$, $D = 0.25$, $\lambda = 1.0$.

overall strength of the nonlinear term is increased by a factor of 4 (these results are not shown) and find the same qualitative behavior. We therefore find excellent agreement between the exact results and those of AGREE.

In Fig. 3 we present the means and variance for the same system but now driven not only by the stochastic term $f(t)$ but also by a harmonic driver $Q(t) = \cos \Omega t$. The agreement is again seen to be excellent for this nonstationary system.

By changing the sign of the linear term in (2.1), i.e., $\lambda \to -\lambda$, we observe in Fig. 4 that the average system response is a monotonically increasing function of time. Here the ensemble calculation is quite close to the AGREE result even with only 100 members in the ensemble. The variance as calculated by AGREE in this case is only accurate for some initial conditions; i.e., the system is initially not near an unstable steady state. As indicated in Section 2.3, an accurate representation for all initial conditions would be a bimodal distribution.[24] A similar behavior has also been observed using a different approximation technique in discussing transient phenomena in the decay of unstable states. We conclude that any approxi-

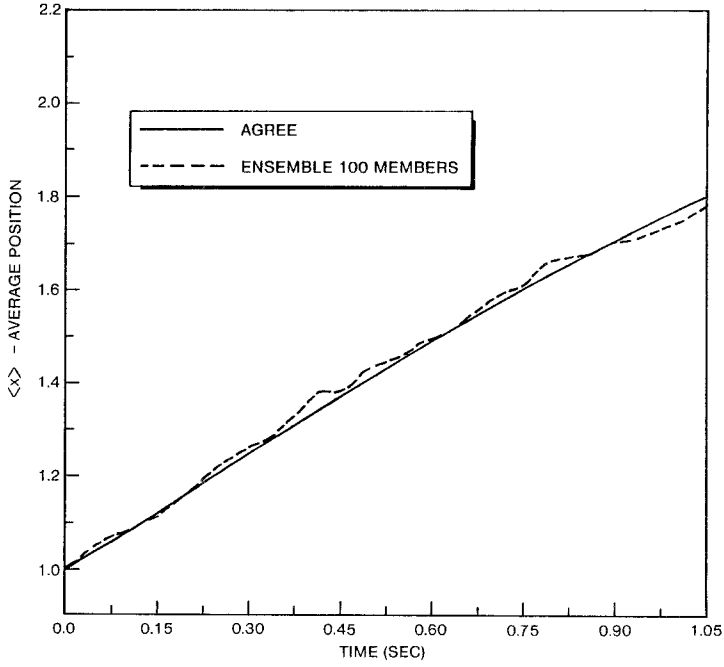


Fig. 4. Same as Fig. 2, but with $\lambda = -1$.

mation scheme using a unimodal distribution for an actually bimodal one would suffer the same defect at long times. An approximate bimodal distribution is more appropriate in this case and would rectify the disagreement.

4. CONCLUSIONS

It is apparent from the calculations that the strategy of AGREE has indeed been successful. The nonlinear stochastic differential equation has been well represented by an "equivalent" linear one whose coefficients are allowed to change with time. This procedure enables us to obtain transient

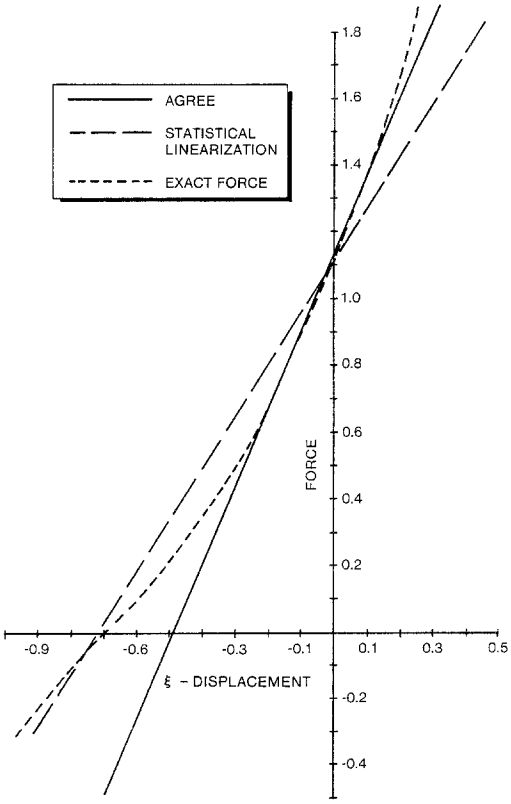


Fig. 5. The gradient of the potential function, i.e., the force, in the statistical linearization and AGREE approximations are compared with the exact one at $t = 0.2$ sec, with $A = B = 0.5$, $\lambda = 1$ and $Q = 0$.

properties that one cannot reliably calculate using ordinary statistical linearization. This can be appreciated by contrasting the evolution of the "equivalent potential" (2.22) with the harmonic potential depicted in Fig. 1. The comparison of the gradient of these two potentials, i.e., the forces at a given instant, is shown in Fig. 5. Given the success of our results, we anticipate that one can successfully model nonlinear stochastic systems with many degrees of freedom in an analogous way.

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